Trigonometry Teacher's Edition - Common Errors

CK-12 Foundation

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Chapter 1

Trigonometry TE - Common Errors

1.1 Trigonometry and Right Angles

This Trigonometry Teaching Tips FlexBook is one of seven Teacher's Edition FlexBooks that accompany the CK-12 Foundation's Trigonometry Student Edition.

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Basic Functions

In-Text Examples

1) Students may get mixed up and think that a relation is a function if it has a unique $x-$value for every $y-$value instead of the other way around. Stress that if the same $x-$value shows up twice, paired with two different $y-$values, then the relation is not a function (so relation A is not), but if every $x-$value has only one corresponding $y-$value, the relation is a function (so relation B is).

2) On part a, students may think the domain is restricted to positive numbers; explain that that restriction applies to the range, not the domain. (We can find the square of any real number x, but the $y-$value we get back will always be positive.)

On part c, students might think of trying to draw a line or curve connecting those points, or might just interpolate to assume that the domain includes all the numbers between 2 and 5. Explain if necessary that those points aren't representative samples of the function, they are all the points in the function, so the domain consists only of the three $x-$values given.

4) Students are likely to try plugging the numbers 15 and 30 into their equations; this will lead to error because those are the times in minutes, not hours, while the speeds given are in miles per hour. Remind them to convert minutes to hours before solving the problem.

6) Of course, the most common mistake here is to forget that only three sides of the enclosure need to be fenced because the barn will make the fourth side.

Review Questions

1) Part a is subject to the same error as Example 1 above. For students stuck on part b, suggest drawing a graph, or just ask them to consider whether it seems possible to plug in the same $x-$value twice and get

two different $y-$values.

3) Part d is a possible sticking point, as students may not know what to do with the equation from part c. Remind them that they need their profit to be greater than zero, so they must find the smallest value of x that will make $P(x)$ greater than zero. Also, remind them that their answer should be a whole number.

4) On part b, students may jump to the conclusion that the range is just the positive real numbers, because the equation resembles $y = x^2$. When they graph the function, point out to them that part of the graph is below the $x-$axis, so the range actually includes those $y-$values as well. Specifically, since the vertex has $y-$coordinate -3.25, the range consists of all real numbers greater than or equal to -3.25.

5) Students may identify the vertical asymptote at $x = -3$ but miss the horizontal one at $y = 1$, especially because it's not as easy to derive from the equation.

6) Students may get confused and set up the equation as if the cost per person were \$500 and they were trying to figure out the total cost for p people. Stress that \$500 is the total cost and that they are trying to figure out how much each person will pay if the bill is split among p people.

Angles in Triangles

In-Text Examples

1) Students shouldn't get hung up on trying to figure out if the triangles are right, acute, or obtuse; it's harder to figure those out from the side lengths alone, but easy to tell if the triangle is equilateral, isosceles, or scalene. They may notice that $3-4-5$ is a Pythagorean triple (making that triangle a right triangle), and it may be worth pointing out that all equilateral triangles are acute (because all three of their angles must measure 60°).

(If they really want to know how to tell if a triangle is acute or obtuse just from the side lengths, you can tell them this: Label the longest side c and the other two sides a and b, and then compare c^2 to the quantity $(a^2 + b^2)$. If c^2 is greater than the sum of the other two squares, the triangle is obtuse; if it is smaller, the triangle is acute; and if it is equal, of course, the triangle is right.)

2) Students should avoid jumping to the conclusion on part c that the 50° angle is one of the two angles that are equal—although it could be, it might not be, and so there are two possible solutions.

Review Questions

1) This problem has two right answers, so don't let students agonize over which one is correct.

4) For part a, students may need to be reminded what a complement is. (Complementary angles are angles that add up to 90°.) Then, they may try to solve part b by working out $180 - 90 - 23 = 67$. This method does yield the correct answer, but misses the point of the problem: once we know the two non-right angles are complementary, all we have to do is subtract 23 from 90 to get the missing angle.

5) Drawing a picture may help students who get stuck on this problem, even though the problem is really more algebra than geometry. The important thing they may forget is that they know all three angles must add up to 180°, which means they can set up the equation $D + O + G = 180$, substitute 2D and 3D for O and G respectively, and solve for D.

8) The triangles certainly look similar, and students may think they are because they have two sides in proportion and one angle the same. However, since the angle isn't between the two sides, we can't actually tell if the triangles are similar.

9) The fact that the numbers 100 and 20 appear next to each other may tempt students to set up the proportion $\frac{20}{100} = \frac{24}{x}$, or $\frac{100}{20} = \frac{24}{x}$. Have them draw a diagram to see which distances they are actually

comparing, or simply remind them that they must pair the flagpole with *its* shadow and the building with its shadow to get the right proportion: $\frac{20}{24} == \frac{x}{100}$.

10) Make sure answers to this problem demonstrate an understanding that similar triangles are not necessarily congruent—i.e. they do not necessarily have the same side lengths.

Measuring Rotation

In-Text Examples

1) For students who have trouble keeping the terms "acute" and "obtuse" straight, the mnemonic "a cute little angle" may help remind them that acute angles are the smaller ones.

2) Since protractors like the one shown here display two different numbers for the measure of an angle, students may read off the wrong one. Remind them to think about whether the angle appears to be acute or obtuse, and figure out which number makes sense based on that. (Another way to check is to note that the end of the protractor they placed against one side of the angle either reads 0° on the inside "track" and 180° on the outside track, or vice versa. The track on which it reads 0° is the one from which they should read off the measure of the other side of the angle. For example, in the illustration shown in the text, the end of the protractor placed against the bottom side of the angle reads 0° on the inside track, so the inside track is the one to use for determining the angle measure, and so we know it is 50° and not 130°.)

3) Students may get their calculations backwards here, possibly due to a vague notion that the larger wheel should rotate a greater number of times. Explain if necessary that since the wheels both travel the same distance along their circumference and the larger wheel has more circumference, it doesn't have to make as many rotations to travel that distance. (A possibly useful analogy is that of a shorter person having to take more steps to keep up with a taller person.)

Review Questions

3) On part c, students may think they are done when they have converted the decimal portion to minutes and forget about converting the remainder to seconds, or they may "convert" to seconds by just copying the number after the decimal point—e.g., expressing 57.6' as 57'6". Remind them that 57.6' is equal to $57\frac{6}{10}$ minutes, and they need to figure out how many seconds are in $\frac{6}{10}$ of a minute if each second is $\frac{1}{60}$ of a minute.

4) Possible errors here include converting the seconds but not the minutes to decimal form, or the minutes but not the seconds. Referring back to page 33 should help students remember how to perform these conversions.

7) Students may get the diameter of the wheels mixed up with the distance between them, and plug in the wrong one at the wrong time.

9) $-120°$ is a possible wrong answer for part a. Demonstrate that an angle of $-120°$ falls in quadrant III, while an angle of 120° falls in quadrant II, so they are not co-terminal.

10) The length of the axles is a red herring, and so is the distance between them; students may think they need to find which one of the four wheels makes the most rotations, but really they only need to find whether the front or back wheels rotate more. (Also, the answers given in the text are incorrect; the numbers of revolutions should be 200 and 66.67 respectively, and the difference in the number of degrees should be 48000.)

Additional Problems

1) What is the angle between the hands of a clock at 6:30? (Remember, the hour hand is not directly on the 6.)

2) Name an angle that is coterminal with -180°.

Answers to Additional Problems

1) 15°

2) Answers will vary. Possible answers include 180° and 540°.

Defining Trigonometric Functions

In-Text Examples

3) Students may mix up the definitions of secant and cosecant. Emphasize that secant is the reciprocal of *co*sine and *co*secant is the reciprocal of sine, so there is exactly one "co-" function per pair of reciprocals (and since tangent and cotangent are reciprocals, they too fit this pattern).

5) Students who are still thinking in terms of angles in triangles may get stuck here. Remind them to think of the trig functions as ratios of $x-$ and $y-$coordinates instead; using the definitions above, they can plug in any values for x and y, even values for which it isn't possible to draw a triangle and measure the side lengths.

Also, because students first learned the definition of sine before the definition of cosine, they can easily get confused and think that the sine value is the $x-$coordinate and the cosine value is the $y-$coordinate. Even if they "know" that's wrong, it's still an easy trap to fall into any time they're not thinking very hard about it (sometimes even after they've been studying trigonometry for quite a while!). Remind them to watch out for this error and to double-check their answers whenever they are finding sines and cosines with this method.

Review Questions

6) The answer to part b, of course, is not simply $2y$; it's true that the length of BD is $2y$, but the point is that it is also 1 because triangle ABD is equilateral. Similarly, the answer to 6c is not just y, but $\frac{1}{2}$; this means that $y = \frac{1}{2}$, and that's important for solving the rest of the problem.

8) The ratios for 60° angles are easy to mix up with the ratios for 30° angles, especially since the values of a given trig function for a 60° angle is the same as the value of the corresponding "co-" function for a 30° angle.

9) Students may give the knee-jerk answer "quadrants I and II" because that's where the $y-$values are positive, or "quadrants I and IV" because that's where the $x-$values are positive. Remind them that the value of the tangent function depends on both the $x-$ and $y-$value: since the tangent is $\frac{y}{x}$, is it positive or negative when x and y are both positive? Both negative? How about when one is positive and the other is negative? In which quadrant(s) does each of those conditions hold?

10) Possible wrong answers include "it's five times 30°," "it's the supplement of 30°," and "it's 30° plus 120°." Although these are all technically true, they aren't what we're looking for because they aren't useful in this case. The correct answer is along the lines of "it's like a 30° angle, but reflected across the $y-$axis," because noticing this fact helps us figure out what the ordered pair for a 150° angle is.

Additional Problems

1) Sketch the angle 210° on the unit circle. What do you think its ordered pair is?

Answers to Additional Problems

1) $\left(\frac{-\sqrt{3}}{2}, \frac{-1}{2}\right)$

Trigonometric Functions of Any Angle

In-Text Examples

1) Make sure students remember that the reference angle is always the distance to the closest part of the $x-$axis, never the $y-$axis—even if the $y-$axis is closer.

4) Students may be momentarily thrown by part b; remind them that an angle of *negative* 300° is co-terminal with an angle of positive 60°.

5) Some students may try dividing the given angles by some number, instead of subtracting 360° from them. Also, on later problems they may forget that they aren't done when they reduce the angle down to one that's less than 360°, and that they still have to find the reference angle for that angle.

6) Note that the cosine column comes before the sine column; this may confuse students momentarily.

7) Make sure calculators are in degree mode; in radian mode the answer will appear to be 0.8589.

Review Questions

7) "Between 10 and 15 degrees" or "between 165 and 170 degrees" is as precise as the answer needs to be. (Either of those approximations is correct, and students should be made aware of this, as the fact that there is more than one angle for a given sine value will be important later.)

8) Students may think of choosing 60° as the "special angle." The value of $\tan(50^\circ)$ is fairly close to the value of $\tan(60^\circ)$, but you'd need a calculator to figure out the value of $\tan(60^\circ)$ in decimal form in order to compare the two tangent values. The value of $\tan(45^\circ)$, though, is simply 1, and in any case 50° is closer to 45° than to 60°, so it makes more sense to use 45° as the special angle.

9) Leaving calculators in radian mode will yield the wrong answers -0.9820 and 45.1831.

10) Students may end up thinking a little too hard about this problem. All they're supposed to conjecture is that the two expressions are not equal, so if any of them struggle with this problem, find out if they've got that much figured out and reassure them they can stop there.

Additional Problems

1) Use a calculator to find the tangent of $86^\circ, 87^\circ, 88^\circ$, and 89°. Then, find the tangent of $94^\circ, 93^\circ, 92^\circ$, and 91°. Now make a conjecture about the behavior of the tangent function as x approaches 90°.

Answers to Additional Problems

1) The values of the tangent function are as follows:

X	$\tan x$
86°	14.3007
87°	19.0811
88°	28.6363
89°	57.2900
90°	undefined
91°	-57.2900
92°	-28.6363
93°	-19.0811
94°	-14.3007

The tangent function approaches infinity as x approaches $90°$ from below, and approaches negative infinity as x approaches $90°$ from above.

Relating Trigonometric Functions

In-Text Examples

4) Demonstrating that $\cot\theta = \frac{\cos\theta}{\sin\theta}$ because cot is the reciprocal of tan and $\tan\theta = \frac{\sin\theta}{\cos\theta}$ is an equally valid answer.

5) Some students may try to simply subtract $\cos\theta$ from 1 to get $\sin\theta$; others may subtract $\cos^2\theta$ from 1 but then forget to take the square root of the answer.

6) The biggest problem here is that students may simply not have any idea where to begin. Telling them the first step may help, but it may also be easier for them to work the problem out "backwards" instead. When you start with $\cot^2\theta + 1 = \csc^2\theta$, the logical first step is to rewrite $\cot^2\theta$ as $\frac{\cos^2\theta}{\sin^2\theta}$ and $\csc^2\theta$ as $\frac{1}{\sin^2\theta}$; this suggests the idea of also writing 1 as $\frac{\sin^2\theta}{\sin^2\theta}$ and then dividing through by $sin^2\theta$.

Review Questions

2) You may need to remind students here not to use the "$\sin^{-1}$" function on their calculators to find the cosecant, but instead to find the sine and then use the "x^{-1}" key to find the reciprocal of the sine.

3) On this and the previous problem, some students may still be confusing the domain with the range. They may also not realize that an input value that makes a function undefined is an input value that must be excluded from the function's domain.

8) Students may get the functions here mixed up with their reciprocals, or may get the Pythagorean identity backwards.

Additional Problems

1) If $\cos\theta = \frac{24}{25}$, what is the value of tan θ?

2) If $\sin\theta = \frac{5}{13}$, what is the value of cot θ?

Answers to Additional Problems

1) $\frac{7}{24}$. Solving this problem takes two steps: first finding $\sin\theta = \frac{7}{25}$ using the Pythagorean identity, and then finding $\frac{\sin\theta}{\cos\theta}$.

2) $\frac{12}{5}$. The problem is similar to the previous one, but students must remember that $\cot\theta$ equals $\frac{\cos\theta}{\sin\theta}$ and not $\frac{\sin\theta}{\cos\theta}$.

Applications of Right Triangle Trigonometry

In-Text Examples

1) Some students may have a hard time understanding which ratios to use to solve which triangles. Try to make clear to them that they should pick a ratio for which they know the angle measure and one of the two sides involved, and then use the ratio to find the other side—or pick a ratio for which they know the two sides, and use the ratio to find the angle. A trig ratio for which they only know one of those three pieces of information won't do them any good, and a ratio for which they already have all three will only tell them what they already know.

2) Using the sine to find the second leg might seem like a good idea, but it would involve plugging in the value we just found for the hypotenuse, which is an approximation. Using the tangent is better because it allows us to plug in the exact value we were given for the length of the first leg, and plugging in an exact value instead of an approximation will yield a more accurate result.

Review Questions

1) As in example 2, students should try to use the given numbers whenever possible rather than plugging the approximate values they've found earlier into later parts of the problem. In this case, that means they should use the sine to find b and the cosine to find a, rather than finding one of them and then using the tangent or the Pythagorean Theorem to find the other.

3) Note that this problem refers to Example 2 in the earlier part of the lesson, not to either of the problems directly above.

7) Students may try to plug in $100°$ as one of the angles in the triangle, when the relevant angles are actually $80°$ and $10°$.

8) Some students may not see what this problem has to do with solving triangles; others may see that they have to divide the quadrilateral into triangles, but may pick the wrong way to do it—drawing a diagonal from upper right to lower left instead of vice versa, which makes the problem unsolvable using the tools they currently have.

9) Don't let students get hung up on trying to figure out how to find the width of the pond at its widest point; "how wide" here just means "how far is it from A to B?"

10) Students may jump to the conclusion that $\triangle PAN$ is a right triangle and try to find x based on the sine or tangent of $50°$.

Additional Problems

1) a) What value would you get for the height of the tree in Example 3 if you did not take the height of the person into account?

b) In example 4, why couldn't we just add 5 feet to the answer we found in the first part (where we didn't take the person's height into account) to get the answer to the second part (where we did)?

Answers to Additional Problems

1) a) 15.63 feet.

b) As you can see from the diagrams, taking the height of the viewer into account in example 3 just required us to shift the triangle up 5 feet. In example 4, however, accounting for the person's height required us to actually lengthen one of the sides of the triangle by 5 feet, which resulted in the proportions of the whole triangle being different. This is why drawing accurate diagrams is important in trigonometry; sometimes we need them to make it absolutely clear what information we can just assume and what information we can't.

1.2 Circular Functions

Radian Measure

In-Text Examples

One very common mistake to make when working in radians is to forget from time to time that a complete rotation is 2π radians, not π radians. This doesn't happen often when one is thinking of a whole circle, but it is very easy to think of a quarter of a circle as being $\frac{\pi}{4}$ radians instead of $\frac{\pi}{2}$, a sixth of a circle as being

$\frac{\pi}{6}$ instead of $\frac{\pi}{3}$, and so on. Students may make this type of error frequently, and may continue making it for quite some time, so remind them more than once to be particularly careful about checking their angle measures when working in radians.

(As evidence that not only students are prone to this particular error, see the illustration to example 4 in the lesson immediately following this one.)

Another common error is to get the degrees-to-radians formula and the radians-to-degrees formula mixed up. The note at the bottom of page 101 should help students with this, though: basically, they should multiply by $\frac{\pi}{180}$ when they want π in their answer (that is, when converting to radians), and should multiply by $\frac{180}{\pi}$ when they have a π to get rid of (that is, when converting from radians to degrees). Of course, there won't always be a π involved when working in radians (see problem 5 below, for example), but the mnemonic will remind them which way the conversion goes if they don't take it too literally.

Review Questions

6) This problem should read "sine" instead of "cosine"; students may therefore think that Gina's typing in "sin" instead of "cos" is the problem, instead of noticing that the calculator is in the wrong mode. They may also get $\frac{-\sqrt{3}}{2}$ instead of $\frac{-1}{2}$ as the actual value, and shouldn't be penalized for this.

Additional Problems

1) What is $\sqrt{2}$ radians in degrees? (Round to the nearest tenth.)

2) What is 90 radians in degrees, and what is its reference angle?

3) What is π degrees in radians? (Round to four decimal places.)

Answers to Additional Problems

1) 81.03°

2) 5156.6°, which is coterminal with 116.6°, so its reference angle is 63.4°.

3) 0.0548.

Applications of Radian Measure

In-Text Examples

1) The most common mistake to make here is to forget that the hour hand is not right on the 11, but a third of the way between 11 and 12. Students who forget this may jump to the conclusion that the hands are $\frac{5}{12}$ of the circle apart, and then may also forget (as mentioned above) that $\frac{5}{12}$ of the circle does not equal $\frac{5\pi}{12}$ radians.

2) A common error has actually been made in the text here: the numbers 12 and 11.81 have been substituted for r in the arc-length formula when each of those numbers is in fact a diameter and not a radius. Students should be cautioned against making the same error on future problems, but should probably be forgiven for making it here.

4) As mentioned earlier, another common error appears in the in-text illustration here: $\frac{2\pi}{3}$ would of course be $\frac{1}{3}$ of the circle, not $\frac{2}{3}$. This will only lead to student error, though, if students try finding the area of the whole circle first and then taking $\frac{2}{3}$ of it instead of $\frac{1}{3}$ of it. If they apply the formula in the text instead, they will not go wrong, since the formula is not based on the illustration.

Review Questions

1) On part a.iii, students should not try to convert the approximate angle measure they found in part a.ii

to degrees (as they may try to do with their calculators, especially since they have just used them to find that approximation). Instead they should start with the exact angle measure from part a.i, and convert it by hand using the formula they learned earlier.

3) Simply miscounting the dots may lead to error here. (There are 32.) Also, in finding the distance between the two dots selected in part b, students may include the dots at both the beginning and the end and conclude that the distance between them is $\frac{14}{32}$ of the circle when it is really $\frac{13}{32}$.

4) Students may forget to take into account both the radius of the outer circle and the radius of the inner circle when calculating the area of each section.

5) This problem is a particularly easy place to make the routine error of plugging in the diameter in place of the radius.

Circular Functions of Real Numbers

Review Questions

1) A quick hint to use similar triangles should help students who get stuck on this problem.

3) Students may get a little confused about where to do the labeling; point out if necessary that the largest circle in the diagram is the unit circle. The other circles are just there to give them a convenient place to write the angle measures without having to write them right on top of the coordinates of the points; if they get mixed up and write the radian measures on the inner circle and the degree measures on the middle circle, there's no need to penalize them as long as they've matched the correct degree and radian measures with the correct angles.

4) Perceptive students may draw the cosine segment in a different place than the book suggests; they may draw it as a horizontal segment extending from the $y-$axis to the point where the sine segment meets the unit circle. This isn't an error; in fact, drawing the segment this way makes it easier to see that the sine and cosine segments have a relationship similar to the relationship between the tangent and cotangent or secant and cosecant segments.

(However, drawing it the standard way is fine too, and probably easier for most.)

5) The correct answer to this question is actually any combination of $\sin, \tan$, and sec.

6) Some students may get the idea that answers b and d must both be true if either one of them is true—that the tangent must get infinitely large when the cotangent gets infinitely small, because they are "opposites" in a sense. The ambiguity of the phrase "infinitely small" doesn't help; it can be taken to mean "approaching zero," but in this case it really means "approaching negative infinity." When the tangent approaches infinity, the cotangent approaches zero, not negative infinity; when the cotangent approaches negative infinity, the tangent approaches zero, and this latter case is what happens when x increases from $\frac{3\pi}{2}$ to 2π. (Referring to the graphs earlier in the lesson will confirm this.)

Additional Problems

1) Why does it make sense that the ranges of the secant and cosecant functions include all numbers except those between 1 and -1? (Hint: think in terms of sine and cosine.)

Answers to Additional Problems

1) The sine and cosine functions only take values between 1 and -1—in other words, only numbers whose absolute value is less than or equal to 1. The secant and cosecant are the reciprocals of the sine and cosine, and the reciprocal of a number whose absolute value is less than or equal to 1 will have an absolute value greater than or equal to 1. (For example, think of a fraction between 0 and 1: its numerator is less than

its denominator, so its reciprocal will have a numerator greater than its denominator and so will be greater than 1.) So, the secant and cosecant functions can only take values whose absolute value is greater than or equal to 1.

Linear and Angular Velocity

In-Text Examples

1) A possible overcomplication of this problem is to think that 15 feet is the "length" of the oval shape formed by the track (i.e. the major axis of an ellipse), rather than simply the track's circumference and thus the distance the car travels per circuit.

2) Students may forget to convert their final answer from miles per minute to miles per hour. If they do remember, they are likely to try to divide the miles per minute result by 60, instead of multiplying it by 60, to get the answer in miles per hour. Encourage them to slow down and think about what the answer really means: if Lois goes .047 miles in one minute, then how far will she go in 60 times one minute?

Review Questions

1) Careless reading may lead students to jump to the answer $\frac{7}{9}$ cm/sec. Remind them that the speed of the dial is the circumference, not the radius, divided by 9 seconds.

3) Note that Doris' horse is not 7 m from the center, but rather 7 m farther from the center than Lois', meaning it is 10 m total from the center.

4) We haven't covered scientific notation in a while, so students may be prone to make mistakes with it. Watch for answers that are off by one or two orders of magnitude.

Additional Problems

1) Two gears mesh with each other so that they rotate in opposite directions, with both their outer edges moving at the same linear velocity. The radius of the larger gear is 10 cm and the radius of the smaller gear is 6 cm. The smaller gear makes two revolutions per second.

a) What is the angular velocity of the smaller gear?

b) What is the linear velocity of a point on its outside edge?

c) What is the angular velocity of the larger gear?

d) How many revolutions does the larger gear make per second?

e) A peg is attached to the larger gear at a point 2 cm from its outer edge. What is the peg's linear velocity?

Answers to Additional Problems

1) a) 4π radians/sec, or approximately 12.57 radians/sec.

b) 24π cm/sec, or approximately 75.40 cm/sec.

c) The linear velocity of a point on its outside edge is the same as that of the smaller gear, or 24π cm/sec, so its angular velocity is 2.4π radians/sec, or approximately 7.54 radians/sec.

d) 1.2 revolutions per second.

e) The peg is 8 cm from the center of the gear, so its linear velocity is 19.2π cm/sec, or approximately 60.32 cm/sec.

Graphing Sine and Cosine Functions

In-Text Examples

2) Students may think the period is 2 units, because each "portion" of the graph is 2 units wide. Explain that it takes one "high" portion and one "low" portion to make up one complete cycle of the graph.

3-6) Despite the explanations in the text, it is likely that some students will still habitually mix up the period with the frequency. Repeated drilling may be the only way to fix this.

Review Questions

2) A few students may be tripped up by part d; since there are no minimum and maximum values, they may get the idea they're supposed to be looking for something else, and supply an answer like "$\frac{\pi}{2}$ and $\frac{-\pi}{2}$" because those are the beginning and ending $x-$values of one cycle of the graph.

The error they are most likely to make on problems like e and f is to think that the number within the parentheses affects the maximum and minimum $y-$values. However, solving problems a through d first should help reinforce that it is only the multiplier out front that matters.

3) Students may try to simply divide both sides by $\sin(x)$; they will then end up with the equation $4 = 1$, for which there is no solution. However, this technique is incorrect because $\sin(x)$ sometimes equals zero on the interval of $x-$values given for the problem, and dividing by an expression that can equal zero often eliminates possible solutions.

Graphing the two functions instead (or simply thinking about them in the right way) will show that whenever $\sin(x)$ equals zero, $4\ \sin(x)$ also equals zero and so the two expressions are equal. This happens at three places on the given interval, including the interval's endpoints.

A more appropriate application of algebra will yield this solution as well: instead of dividing both sides by $\sin(x)$, subtract $\sin(x)$ from both sides to yield $3\ \sin(x) = 0$, and then divide by 3 to get $\sin(x) = 0$.

4) Getting the period and frequency mixed up is the most likely error here; reviewing the definitions may help.

5) Students may get twice the correct value for the amplitude here, forgetting that it's the distance from the minimum or maximum to the *middle* of the graph rather than to the maximum or minimum. Also, when writing out the equation, they may again get mixed up about which number goes where.

6) At this point, students may still be stretching the graph when they should be shrinking it, or vice versa.

Translating Sine and Cosine Functions

In-Text Examples

1) A few students may think we're still dealing with amplitude here, and that the maximum and minimum are 6 and -6. Remind them that we are now shifting the graph rather than stretching it; the maximum and minimum remain the same distance apart, but have both moved together 6 units from where they started out.

Review Questions

1-5) Starting with the questions and trying to match them to the functions right away is tempting, but almost certainly the wrong way to approach this set of problems; it's much better to sketch out graphs of the functions and then match them to the appropriate descriptions.

Also, the questions about $y-$intercepts may be confusing, as we haven't directly discussed those with regard to trig functions specifically. The $y-$intercepts can't be derived directly from the amplitude, frequency, period, phase shift, or vertical shift, as students may be tempted to try, but they can often be read off the graph fairly easily. The best way to find them, however, is simply to plug in 0 for x and then calculate the value of y. (Remember, the $y-$intercept is simply the value of y when the graph crosses the $y-$axis, which is to say when x equals 0.)

6) "Express the equation as both a sine and cosine function" may be misconstrued to mean "write one equation that involves both sine and cosine" rather than "write one equation that involves sine and another that involves cosine."

Also, watch out for students getting the sine and cosine graphs mixed up—not just on this problem, but on any problem that involves phase-shifted sinusoids (like the next four problems!).

7-10) Students may unfortunately be thrown by the fact that graphs A and C are incorrectly drawn: they are shifted up 2 units when they should be shifted 1 unit. Explain this to avoid confusing them.

Students may also still be getting mixed up about whether a shift to the left (or to the right) should be described with a negative or positive number, and this is a particularly bad time to make that error because it may be compounded with the error of mixing up sine and cosine graphs, resulting in students mistaking a sine graph shifted one way for a cosine graph shifted the other way (or even shifted the same way).

11) It may be hard to tell here that the tick marks on the $x-$axis represent 1 unit each, rather than π or $\frac{\pi}{2}$ units. (The multiples of π are indicated in approximately the right positions, but without tick marks.)

General Sinusoidal Graphs

In-Text Examples

1) On problems like this, students may still be getting mixed up about which transformations correspond to which parts of the equation. In particular, they may confuse the amplitude with the vertical shift and the frequency with the phase shift, or get the frequency and the period mixed up.

3) One knee-jerk error here is to assume that the amplitude is the same as the maximum value, or in this case 60. (In reality, the amplitude is indeed the same as the maximum value whenever the vertical shift is 0, but otherwise the maximum value is equal to the amplitude *plus* the vertical shift—in this case, the vertical shift is 20, so the amplitude is 40.)

Another common error is, once again, to forget that the amplitude is only half the total height of the graph.

Review Questions

1-5) Trying to find the maximum and minimum after finding the amplitude but before finding the vertical shift may lead students to get the wrong values, as they may default to treating the graph as if it were centered at 0.

Additional Problems

1) The graph of $y = 2 + \cos(3(x - \pi))$ is translated an additional $\frac{\pi}{2}$ units to the left and 3 units down. What is the equation of the new graph?

2) The graph of $y = 3 + 2\sin\left(6\left(x - \frac{\pi}{3}\right)\right)$ is stretched so that its period is twice as long.

What is the frequency of the new sine wave?

3) Which of the following yields the same graph as $y = 2 + \cos\left(2\left(x + \frac{\pi}{2}\right)\right)$?

a) $y = 2 + \cos\left(2\left(x - \frac{\pi}{2}\right)\right)$

b) $y = 2 + \sin\left(2\left(x - \frac{\pi}{2}\right)\right)$

c) $y = 2 + \sin\left(2\left(x + \frac{\pi}{2}\right)\right)$

d) $y = 2 - \cos\left(2\left(x + \frac{\pi}{2}\right)\right)$

Answers to Additional Problems

1) $y = -1 + \cos\left(3\left(x - \frac{\pi}{2}\right)\right)$

2) If the period is doubled, the frequency is halved. The old frequency is 6, so the new frequency is 3.

3) a

1.3 Trigonometric Identities

Fundamental Identities

Review Questions

1-3) Students may still not quite understand that they need to narrow down what quadrant the angle is in before finding the other trig functions; instead they may just assume it is in the first plausible quadrant they think of.

5) The values of sine and cosine are reversed in the answer key; the sine should be $\frac{-4}{5}$ and the cosine should be $\frac{3}{5}$.

7) Students may have a hard time figuring out where the θ they are supposed to be dealing with comes from. They may need to be walked through the first couple steps of drawing the given triangle and picking one of its angles to be θ so that they can find the sine and cosine of θ and go from there.

8) Students may not immediately see that part a is simply a difference of two squares, or they may not remember the formula for factoring a difference of squares. Thinking of $\sin\theta$ as "x" and $\cos\theta$ as "y" may make the problem look more familiar, and thinking of $\sin\theta$ as "x" will definitely make part b easier.

9) This is one of those fractions that will tempt students to try canceling terms that really don't cancel; they may end up thinking the whole thing can be reduced to $\sin^2\theta - \cos^2\theta$ and then trying to go from there. The actual solution, as explained in the text, is a bit tricky, as it involves factoring the numerator as a difference of squares, as they did in problem 8, but not factoring the denominator. Students who try factoring the denominator as well aren't doing anything mathematically wrong, but they will have to go through a couple of extra steps as a result.

10) Students who remember the proof in the last chapter may try to use segments on the unit circle for this proof. This isn't technically wrong, although the real point here is for them to see that they can prove the identity by expressing it in terms of sine and cosine.

Additional Problems

1) $\sin\theta = \frac{2}{3}$ and $\tan\theta = -3$. What is the value of $\cos\theta$ and what quadrant is θ in?

2) If $\tan\theta = \frac{3}{5}$, what are the possible values of $\sec\theta$?

Answers to Additional Problems

1) $\cos\theta = \frac{-2}{9}$ and θ is in Quadrant II.

2) The Pythagorean identity from problem 10 is the best way to solve this. $\tan^2\theta = \frac{9}{25}$, so $\tan^2\theta + 1 =$

$\frac{34}{25} = \sec^2\theta$, so $\sec\theta = \pm\frac{\sqrt{34}}{\sqrt{25}} = \pm\frac{\sqrt{34}}{5}$.

Verifying Identities

Review Questions

1) Students will get stuck on this problem if it doesn't occur to them to express everything in terms of sine and cosine. You may want to stress that this is almost always a useful technique.

Also, they may get mixed up and express $\sec x$ as $\frac{1}{\sin x}$ instead of $\frac{1}{\cos x}$.

2) There is more than one approach to this problem, so it's actually hard to go wrong; the solution presented in the text is just one way of verifying the identity. However, students may have a hard time figuring out how to begin. The best thing for them to do is just think of any substitution they can usefully make, and then simplify the expression and see what seems useful to do next.

3) Students may try to simply add the denominators or otherwise go the wrong way about finding a common denominator. Also, they may (on this and the next few problems) try to substitute $\sin x$ for $1 + \cos x$ or $1 - \cos x$.

4) Cross-multiplying is the easiest way to solve this problem, but students may not think of that right away because they've been told they should usually only work on one side of the problem at a time. You may want to let them know that when both sides of the equation are fractions (each side must be a single fraction with no other terms), cross-multiplying is often a useful first step.

6) The expression $1 - 2\sin^2 b$ may look as though it can be treated like $1 - \sin^2 b$, which is equal to $\cos^2 b$. $2 - 2\sin^2 b$ would indeed be equal to $2\cos^2 b$, because of the common factor of 2, but $1 - 2\sin^2 b$ doesn't equal anything immediately useful. Simplifying the left-hand side rather than the right is the approach students should take.

7) The multiple negative signs here may lead to sign errors.

8) This is another occasion for misapplying a Pythagorean identity: students may treat $(\sec x - \tan x)^2$ as $\sec^2 x - \tan^2 x$, which equals 1.

Additional Problems

1) Double-check the identity from problem 1 above by verifying that it holds true for $x = \frac{\pi}{4}$.

2) Verify any other identity from the problem set above by plugging in an angle measure of your choice.

Answers to Additional Problems

1) Plugging in $x = \frac{\pi}{4}$ gives us $\sin\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right) = \sec\left(\frac{\pi}{4}\right)$; calculating the trig values yields $\frac{\sqrt{2}}{2} \times 1 + \frac{\sqrt{2}}{2} = \sqrt{2}$, or $2 \times \frac{\sqrt{2}}{2} = \sqrt{2}$, or $\sqrt{2} = \sqrt{2}$. QED.

2) Answers will vary.

Sum and Difference Identities for Cosine

Review Questions

On all the problems here, students may simply mix up the sum formula with the difference formula; the formulas are a little counterintuitive, since the sum formula involves subtracting and the difference formula involves adding.

1) Students will get stuck on this one unless they realize that $\frac{5\pi}{12}$ is the sum of $\frac{3\pi}{12}$ and $\frac{2\pi}{12}$, and that those in turn are equal to $\frac{\pi}{4}$ and $\frac{\pi}{6}$, whose trig values they are already familiar with.

2) Students may think they are done here after they find the cosines of y and z respectively. More commonly, they may think that they need to find the angle measures of y and z so they can find out what $y - z$ is in order to find its cosine—thereby missing the point of the problem, which is that they only need to find the cosines of y and z and then plug those, together with the sines of y and z, into the difference formula for cosines.

3) Some students may try adding up a bunch of first-quadrant angles to get $345°$, forcing them to apply the sum formula multiple times, because they have forgotten that the trig values for key angles in the fourth quadrant are the same as those in the first.

6) Students may try canceling terms before separating the fraction into two fractions.

7) Students may forget that π is an actual angle whose trig values they know, rather than just a variable like θ. They may get $\sin \pi$ and $\cos \pi$ mixed up, or may get them mixed up with $\sin 2\pi$ and $\cos 2\pi$ (that is, $\sin 0$ and $\cos 0$).

8) This is one problem where expressing everything in terms of sine and cosine may actually make things harder; as the solution key shows, it's easier to simply express the left side in terms of tangent.

Another way students may make the problem harder than it needs to be is by cross-multiplying. While that is often a useful technique when dealing with proportions like this, in this case it yields some very complicated expressions that take some work to simplify.

9) After applying the sum and difference formulas to the left-hand side, students are likely to get stuck. The solution key shows the most useful Pythagorean substitutions to make next, but the key insight is simply that Pythagorean identities in general are the tool to use.

10) It's tempting here to use the sum identity first, but it's a much better idea to divide by 2, take the square root of both sides, and *then* apply the sum identity. (And when taking the square root of both sides, we must remember to account for both the positive and negative solutions.)

Some students may also be tempted by the $\cos^2$ term to try using a Pythagorean identity, which will not help at all.

Sum and Difference Identities for Sine and Tangent

Review Questions

1) This is another classic occasion for forgetting that 2π radians instead of π radians make up a circle. Students who have that particular memory lapse in this case may get the idea that $\frac{17\pi}{12}$ is the same as one whole rotation plus $\frac{5\pi}{12}$—especially since the number $\frac{5\pi}{12}$ was just mentioned in the last paragraph. This won't cause their final answer to be wildly wrong—it will just be positive when it should be negative—but because the error seems so minor, they may not be able to figure out where they went wrong.

2) As on problem 3 of the previous lesson, students may try adding up a bunch of first-quadrant angles to get $345°$, forcing them to apply the sum formula multiple times, because they have forgotten that the trig values for key angles in the fourth quadrant are the same as those in the first.

3) As on problem 2 of the previous lesson, students may think they are done here after they find the sines of y and z respectively, or they may think that they need to find the angle measures of y and z so they can find out what $y + z$ is in order to find its sine, rather than simply finding the sines of the angles and then plugging them into the sum formula.

4) Some students may get stuck trying to find the sine and cosine of $5°$ and $25°$ and plug them into the given expression, which there isn't any good way to do given the knowledge they have so far. The trick here, of course, is to recognize that this problem asks them to apply a sum formula backwards; once they see that the given expression is equivalent to $\sin(5° + 25°)$, the problem becomes easy.

5) The right-hand side looks a bit like a Pythagorean identity but isn't; students should focus on simplifying the left-hand side instead of looking for ways to simplify the right.

6) As on problem 7 of the previous lesson, students may forget that π is an actual angle whose trig values they know, rather than just a variable like θ; they may also get $\tan \pi$ mixed up with $\tan 2\pi$ (that is, $\tan 0$).

7) Problems like this, with fractions in the numerator and denominator of other fractions, are a likely place for students to make basic fraction-multiplying errors. Some of them may also still have trouble rationalizing denominators.

10) The same temptation to use the sum identity first may happen here as on problem 10 of the previous lesson.

Additional Problems

1) Find the exact value of $\sin -15°$.

Answers to Additional Problems

1) Using the difference formula: $\sin(-15°) = \sin(30° - 45°)$

$$\begin{aligned}
&= \sin 30° \cos 45° - \cos 30° \sin 45° \\
&= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} \\
&= \frac{\sqrt{2} - \sqrt{6}}{2}
\end{aligned}$$

Double-Angle Identities

Review Questions

1) Students may try to find the measure of angle x when they don't really need to; they'll get stuck if they try to find the trig functions of $2x$ that way instead of by using the double-angle rules.

2) Some students might get this particular double-angle formula mixed up with the Pythagorean identity and think this is a trick question whose answer is simply 1.

3) Treating the triple angle like a double angle is a trap students may fall into. They may also simply get stuck on this problem because the double-angle formulas are the only thing fresh in their minds and they've forgotten that they can also use the sum and difference formulas.

4) Expanding $\cos 2t$ and then simplifying the right-hand side is a tempting first step to try; it's not immediately obvious that this method won't get results as easily as the method outlined in the solution key will.

5) This problem is of course subject to the same error as problem 1.

6) It's very tempting here to start by moving $\sin x$ to the right-hand side of the equation. After doing that and then expanding $\sin 2x$, the next logical step seems to be to divide both sides by $\sin x$, leaving $2 \cos x = -1$. This does yield two of the correct solutions to the equation, but eliminates the other two; dividing both sides

by an expression that can equal zero usually eliminates solutions that shouldn't be eliminated. You may need to remind students of this fact so that they don't do something similar on other problems; in general, dividing both sides of an equation by a trig expression that can equal zero (and most of them can) is not a good idea.

8) The most likely error for students to make here is to think they are done when they get $\frac{1}{4}(\cos^2 2x + 2\cos 2x + 1)$, forgetting that the answer is supposed to be in terms of the first power of cosine (which means they need to get rid of that $\cos^2$ term). Then, they may also be unsure how to convert, or not realize they need to convert, the $\cos^2$ formula so that it works when the argument is $2x$ and not just x.

The same applies to problem 9.

10) This problem shouldn't be too difficult once students understand the previous problems, except that they may not think of expressing $\tan x$ as $\frac{\sin x}{\cos x}$ right away.

Additional Problems

1) Use double-angle identities to verify the values of $\cos\pi$ and $\sin\pi$.

Answers to Additional Problems

1) $\cos\pi = 2\cos^2\frac{\pi}{2} - 1$

$$\begin{aligned} &= 2 \cdot 0^2 - 1 \\ &= 0 - 1 \\ &= -1 \end{aligned}$$

$$\begin{aligned} \sin\pi &= 2\sin\frac{\pi}{2}\cos\frac{\pi}{2} \\ &= 2 \cdot 1 \cdot 0 \\ &= 0 \end{aligned}$$

Half-Angle Identities

Review Questions

1) Students might forget that 225° is an angle they should be familiar with, although hopefully Example 1 earlier in this section will have reminded them.

In general, students may have a hard time recognizing the angles they get when they double the angles given in the first few problems as familiar ones—but since the lesson is about half-angle identities, they should at least have the general idea that doubling the angles in the problems might be useful, and then be motivated to think a little harder about whether they do in fact know the trig functions for those doubled angles.

However, they may also not realize at first that they should double the angles; it's easy to get the notions of "half angles" and "double angles" mixed up, and the fact that one has to *double* the original angle to get the right values to use in the *half*-angle formula, and vice versa, can be a bit confusing.

5) As in problem 1 from the previous lesson, students may need to be discouraged from trying to find the actual measure of angle θ. They have enough information to find the value of $\cos\theta$, and they should do that and go from there. The same applies to problem 8 below.

6) The process for solving this problem involves a lot of fractions within fractions; it's quite easy to mess up here by flipping fractions that shouldn't be flipped, or multiplying fractions that should be divided, or vice versa. Some students may also still be mixing up secant and cosecant, especially when there are this many of both floating around.

Additionally, students are likely to try to solve the problem by simplifying one side only, or by simplifying just one side at a time, which has worked on many problems in the past but won't really work here. After simplifying the right-hand side a lot and the left-hand side a little, they will need to start working on both sides at once. The same applies to problem 7.

9) Students may try applying the half-angle identity before isolating $\cos \frac{x}{2}$; this isn't technically wrong, but may overcomplicate the problem and cause them to make more errors.

Additional Problems

1) Use half-angle identities to verify the values of $\sin \pi, \cos \pi$, and $\tan \pi$.

Answers to Additional Problems

1)

$$\begin{aligned}
\sin \pi &= \pm \sqrt{\frac{1-\cos 2\pi}{2}} \\
&= \pm \sqrt{\frac{1-1}{2}} \\
&= \pm \sqrt{\frac{0}{2}} \\
&= \pm \sqrt{0} \\
&= 0
\end{aligned}$$

2)

$$\begin{aligned}
\cos \pi &= \pm \sqrt{\frac{\cos 2\pi + 1}{2}} \\
&= \pm \sqrt{\frac{1+1}{2}} \\
&= \pm \sqrt{\frac{2}{2}} \\
&= \pm \sqrt{1} \\
&= \pm 1 \\
&= -1 (\text{because cosine is negative in the second and third quadrants})
\end{aligned}$$

3)

$$\begin{aligned}
\tan\pi &= \pm\sqrt{\frac{1-\cos 2\pi}{1+\cos\pi}} \\
&= \pm\sqrt{\frac{1-1}{1+1}} \\
&= \pm\sqrt{\frac{0}{2}} \\
&= \pm\sqrt{0} \\
&= 0
\end{aligned}$$

Product-and-Sum, Sum-and-Product and Linear Combinations of Identities

Review Questions

1) Some students may have trouble jumping from the abstract to the concrete in order to see how the sum-to-product rule applies here; a few may even try expressing the sum as $\sin 14x$.

6) Converting the equation to $\sin 4x = -\sin 2x$ is one tempting wrong path here, as on problem 7.

10) The sum-to-product formula won't help here.

Additional Problems

1) Express the sum as a product: $\cos 50^\circ + \cos 30^\circ$.

2) Verify the identity $\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$ for $\alpha = \frac{\pi}{3}$ and $\beta = \frac{\pi}{6}$.

Answers to Additional Problems

1)

$$\begin{aligned}
&\cos 50^\circ + \cos 30^\circ \\
&= 2\cos\left(\frac{50^\circ + 30^\circ}{2}\right)\cos\left(\frac{50^\circ - 30^\circ}{2}\right) \\
&= 2\cos\left(\frac{80^\circ}{2}\right)\cos\left(\frac{20^\circ}{2}\right) \\
&= 2\cos(40^\circ)\cos(10^\circ)
\end{aligned}$$

2)

$$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$$
$$\therefore \cos\frac{\pi}{3}\sin\frac{\pi}{6} = \frac{1}{2}\left[\sin\left(\frac{\pi}{3}+\frac{\pi}{6}\right) - \sin\left(\frac{\pi}{3}-\frac{\pi}{6}\right)\right]$$
$$\therefore \cos\frac{\pi}{3}\sin\frac{\pi}{6} = \frac{1}{2}\left[\sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{6}\right)\right]$$
$$\therefore \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{2}\left[1-\frac{1}{2}\right]$$
$$\therefore \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{2}\cdot\frac{1}{2}$$
$$\therefore \frac{1}{4} = \frac{1}{4} Q.E.D.$$

Chapter Review Exercises

Review Questions

2) Some students may try to cancel terms before they factor the numerator.

4) Watch out for students multiplying fractions when they should be adding them, or vice versa.

5) Students thinking they don't know the arcsecant of 2 may try to find it with their calculators and end up with an approximation instead of an exact answer. The trick is to realize that the arcsecant of 2 is the arccosine of $\frac{1}{2}$, which they do know.

8) Some students may stop once they find $2x$, instead of going on to find x.

11-12) It's easy to get the angle sum identities mixed up with the sum-to-product identities; the trick is to keep straight which ones are for a single trig function applied to a sum of two angles, and which ones are for the sum of two separate trig functions. It may be useful, before assigning these exercises, to review all the identities at once and point out the similarities and differences between them.

13) This looks like a case for the sum-to-product identity at first glance, but there isn't actually a sum-to-product identity for the sum of a sine and a cosine; students might get mixed up, though, and try one of the other sum-to-product identities, ending up with a somewhat simplified but wrong answer.

14) This is a tricky problem; starting out by expressing $6x$ as $3x + 3x$ might look like the way to go, but that turns out to add several extra steps to the process, and also adds more terms that are harder to keep track of.

Additional Problems

1) Write as a sum: $\sin(9x) \quad \cos(3x)$

2) Verify that the identity from problem 1 holds true for $x = \frac{\pi}{12}$.

Answers to Additional Problems

1)

$$\sin 9x\cos 3x = \frac{1}{2}[\sin(9x+3x) + \sin(9x-3x)]$$
$$= \frac{1}{2}[\sin(12x) + \sin(6x)]$$

2)

$$
\begin{aligned}
\sin 9x \cos 3x &= \frac{1}{2}[\sin(12x) + \sin(6x)] \\
\sin 9\left(\frac{\pi}{12}\right) \cos 3\left(\frac{\pi}{12}\right) &= \frac{1}{2}\left[\sin\left(12\left(\frac{\pi}{12}\right)\right) + \sin\left(6\left(\frac{\pi}{12}\right)\right)\right] \\
\therefore \sin\left(\frac{3\pi}{4}\right) \cos\left(\frac{\pi}{4}\right) &= \frac{1}{2}\left[\sin(\pi) + \sin\left(\frac{\pi}{2}\right)\right] \\
\therefore \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} &= \frac{1}{2}[0+1] \\
\therefore \frac{2}{4} &= \frac{1}{2} \cdot 1 \\
\therefore \frac{1}{2} &= \frac{1}{2} \quad Q.E.D.
\end{aligned}
$$

1.4 Inverse Functions and Trigonometric Equations

Inverse Trigonometric Functions

General Definitions of Inverse Trigonometric Functions

In-Text Examples

1) Students who don't refer to the illustration may get mixed up about which angle is θ. If they start with $\tan\theta = \frac{16 \text{ feet}}{10 \text{ feet}}$, they will end up with 58° as their answer, which is the angle the boards will make with the short side of the deck rather than the long side.

Students may also be using the wrong key on their calculators to find the inverse tangent; they may be just using the "tan" key by itself, or they may be using a combination of the "tan" key and the reciprocal (x^{-1}) key.

Review Questions

1) Some students will still be confusing the vertical and horizontal line tests here, or otherwise be fuzzy on the idea of what makes a relation a function. (Also, the inverse of relation i is in fact a function.)

2) Many students will try to make side BC in the diagram, rather than AC, equal 9 feet, thinking that the ladder is 9 feet up the wall rather than 9 feet long itself.

Additional Problems

1) While walking home, you decide to take a shortcut across an empty lot. From one corner of the lot, you cannot see the opposite corner, but you know the lot is 30 yards long and 20 yards wide. At what angle should you set off across the lot in order to aim directly for the opposite corner?

Answers to Additional Problems

1) $\tan^{-1}\left(\frac{30}{20}\right) \approx 56.31°$, so you should set off at a 56.31° angle from the shorter side (or a 33.69° angle from the longer side).

Using the "Inverse" Notation

In-Text Examples

1) Students may need a refresher on the unit circle at this point; in particular, they may have forgotten how to figure out what quadrant the reference triangle should be in.

Review Questions

1) This is a trick question; students may get confused by the fact that $\frac{\pi}{2}$ is a commonplace angle and may try to take the sine of it instead of the inverse sine, or they may try to take the inverse sine of it but be confused about how to find an angle whose sine is $\frac{\pi}{2}$ (which they should be, since there is no such angle).

Exact Values of Inverse Functions

Exact Values of Special Inverse Circular Functions

Review Questions

Using special triangles may be the wrong approach for these problems, as the angles here are not acute ones and therefore can't be directly found in triangles. Triangles may still be useful for figuring out the reference angles for the angles given, but that's after students figure out what quadrant the angles are in.

When using the unit circle, though, some students may need to be reminded that they can simply read off the sine and cosine values from the coordinates of the points on the circle, as the sine values is equal to the $y-$coordinate and the cosine value is the $x-$coordinate.

Range of the Outside Function, Domain of the Inside Function

In-Text Examples

1) Students may not realize yet that the easiest way to prove two functions are each other's inverses is to compose the two functions and prove that the composition always equals x; they may instead try to derive the one function from the other by swapping the variables, which will work but is not as easy.

Review Questions

1) Both of these functions are actually not invertible simply because their inverses are not functions.

Applications, Technological Tools

Additional Problems

1) Find the inverse of the following function: $f(x) = 3 + 5\sin(2x - 7)$.

Answers to Additional Problems

1)

$$
\begin{aligned}
f(x) &= 3 + 5\sin(2x-7)\\
\therefore x &= 3 + 5\sin(2y-7)\\
\therefore x - 3 &= 5\sin(2y-7)\\
\therefore \frac{x-3}{5} &= \sin(2y-7)\\
\therefore \sin^{-1}\left(\frac{x-3}{5}\right) + 7 &= 2y\\
\therefore \frac{1}{2}\left(\sin^{-1}\left(\frac{x-3}{5}\right) + 7\right) &= y\\
\therefore f^{-1}(x) &= \frac{1}{2}\left(\sin^{-1}\left(\frac{x-3}{5}\right) + 7\right)
\end{aligned}
$$

Properties of Inverse Circular Functions

Derive Properties of Other Five Inverse Circular Functions in Terms of Arctan

In-Text Examples

1) Remind students to label which angle is θ when they are drawing triangles to solve these problems; otherwise they may end up taking the sine or cosine of the wrong angle.

(Of course, they can simply plug the values of x into the set of equations above to solve these problems without drawing triangles. This method is faster, but may mask a lack of understanding of the actual principles involved.)

2) On part a, students may miss the part about the bottom of the screen not being directly on the ground. On part b, they may be confused by the fact that they don't have enough information to find exact angle measures. Instead, they must find a formula for the angle measure in terms of x, and then figure out what value of x maximizes this angle.

Review Questions

2) After all the previous discussion of how $f^{-1}(f(x)) = x$, students are likely to assume that $\tan^{-1}(\tan x) = x$. The only reason they're wrong is that the tangent function has to have its domain restricted in order to be invertible; inside this limited domain, $\tan^{-1}(\tan x)$ is indeed x, but outside of it, the values of $\tan^{-1}(\tan x)$ just repeat themselves, as students will see if they use their calculators to graph it. (Also, those vertical lines on the graph represent vertical asymptotes; that is, they are not actually places where the graph is vertical, but rather where the graph is undefined.)

Additional Problems

1) Express the function $2\cos(\tan^{-1} 3x)$ as an algebraic expression involving no trigonometric functions.

Answers to Additional Problems

1) $2\ \cos(\tan^{-1} 3x) = 2\ \cos\theta = \frac{2}{\sqrt{(3x)^2+1}} = \frac{2}{\sqrt{9x^2+1}}$

Derive Inverse Cofunction Properties

Review Questions

Remembering to get the domain restrictions right should be the only tricky part here.

Find Exact Values of Functions of Inverse Functions Using Pythagorean Triples

In-Text Examples

3) The most likely error here is getting the quadrant wrong.

Applications of Inverse Circular Functions

Revisiting $y = c + a \ \cos b(x - d)$

In-Text Examples

Students may make the same errors here that they made when they originally covered this topic, particularly getting the various shifts and stretches mixed up and getting the sign of the phase shift backwards.

Additional Problems

1) How would you express the equation from problem 1 above as a transformation of $y = \sin x$?

Answers to Additional Problems

1) $y = 3 \ \sin(2(x + 15°)) + 2$, or $y = 3 \ \sin(2(x - 165°)) + 2$

Solving for Particular Values in Trigonometric Equations

Review Questions

1) Students may still be unsure how to derive an equation from just the two points given. The trick is to realize that the horizontal distance between the two points is half the period (and the frequency is 2π divided by the period); the vertical distance between them is the amplitude; the average of the two heights is the vertical shift; and the horizontal distance between the maximum point and zero (divided by 2π) is the phase shift (if we are using a cosine function to model the graph).

Additional Problems

1) You are riding a Ferris wheel at an amusement park. 15 seconds after the wheel starts turning, you are at the top of the wheel, 100 feet high. 20 seconds later, you are at the bottom of the wheel, 10 feet off the ground. At what time did you first reach a height of 70 feet?

Answers to Additional Problems

1) The equation that models this problem is $55 + 45 \ \cos \frac{\pi}{20}\left(x - \frac{15}{2\pi}\right)$. Solving for x in terms of y gives us:

$$x = \frac{\cos^{-1}\left[\frac{y-c}{a}\right]}{b} + d \quad y = 70, \quad c = 55, \quad a = 45, \quad b = \frac{\pi}{20}, \quad d = \frac{15}{2\pi}$$
$$\therefore x = \frac{\cos^{-1}\left[\frac{70-55}{45}\right]}{\frac{x}{20}} + \frac{15}{2\pi}$$

Using a calculator gives us an answer of approximately 10.22 seconds after the Ferris wheel began moving.

Trigonometric Equations

Solving Trigonometric Equations Analytically

In-Text Examples

1) Students may get stuck on these if they don't remember the basic identities.

2) Make sure to point out on part b that there are a total of four solutions on the given interval.

Review Questions

1) Of course, students are likely to miss the fact that they need to double the interval of possible solutions (as explained in the solution key). Even if they do figure this out, they may not realize that this means the two solutions within the interval $[0, 2\pi)$ must each have 2π added to them to make the other two solutions. And finally, they may forget to divide the solutions by 2 at the end to get the values for θ rather than 2θ.

2) As on other problems, students may forget to look for solutions in more than one quadrant. They may also forget to consider the cosines of both $\frac{1}{4}$ and $-\frac{1}{4}$.

3) The trick here is to treat 4θ as a double angle and 2θ as the corresponding single angle; students may not think of this right away. Also, they may make the same mistakes as in problem 1.

4) The same errors as on problem 1 apply.

Solve Trig Equations (Factoring)

In-Text Examples

2) Students will of course be tempted to divide both sides of the equation by $\tan\ x + 1$; the only problem with this is that it will eliminate possible solutions because $\tan x + 1$ can equal zero. The way to get around this is to find the solutions for $2\ \sin x = 1$, but then also find the solutions for $\tan x + 1 = 0$. The solution presented in the text is another way of doing this.

Review Questions

1) Upon finding that $\sin x$ can equal either 3 or -1, students may be stumped by the 3, since that isn't a possible value for $\sin x$. It doesn't mean they've done anything wrong, it just means they should discard that solution.

2) Again, beware of dividing through by $\tan x$, as this will eliminate the solution $x = 0$.

Solve Equations (Using Identities)

Review Questions

1) Getting the signs reversed when factoring is the most likely error here.

2) Students are likely to forget the "for all values of x" part; they may also forget the principal solution $\frac{3\pi}{2}$.

Trigonometric Equations with Multiple Angles

Solve Equations (with Double Angles)

In-Text Examples

1) Again, beware of dividing both sides by $\cos x$.

4) ...or by $\sin x$.

Review Questions

1) Finding the actual measure of angle x is tempting but not necessary.

2) The solution to part b involves realizing that $\cos^4\theta - \sin^4\theta$ can be treated as a difference of squares, which may escape students at first.

3) This equation is actually true for all values of θ.

Solving Trigonometric Equations Using Half Angle Formulas

In-Text Examples

3) Students may be momentarily stumped by the appearance of the unfamiliar-looking angle $\frac{7\pi}{6}$. Rewriting it as $\pi + \frac{\pi}{6}$ may help them see that it is simply a third-quadrant angle with a reference angle of $\frac{\pi}{6}$, whose sine and cosine they should be able to figure out easily.

Review Questions

3) Students will get three solutions to this equation if they follow the procedure outlined in the solution key, but not all of those solutions will in fact be correct. Squaring both sides of an equation introduces extraneous solutions, so they must check their answers afterward to see which ones need to be discarded. This is true any time both sides of an algebraic equation are squared.

Solving Trigonometric Equations with Multiple Angles

In-Text Examples

2) Once again, remind students that they must look for values of 2θ between 0 and 4π in order to find values of θ between 0 and 2π.

Review Questions

2) Students may get stuck trying to figure out how to express $\cos 3x$ in terms of $\cos x$, perhaps by trying to refer back to the "triple-angle formula" derived in chapter 3, or trying to derive one themselves from the double- and half-angle formulas they already know. This isn't necessary, though; they only need to solve for $\cos 3x$ and then find possible values for $3x$ that lie between 0 and 6π in order to find the possible values of x that lie between 0 and 2π.

3) After finding the principal solutions to the equation, students may try to derive the other solutions by adding $2\pi k$, rather than πk. (If they were dealing with the sine or cosine functions instead of the tangent function, they'd be right, but the tangent function repeats its values every π rather than 2π units.)

Equations with Inverse Circular Functions

Solving Trigonometric Equations Using Inverse Notation

In-Text Examples

2) Students may be momentarily confused by the lack of "arc"-function buttons on their calculators, and may need to be reminded to use the " $\sin^{-1}$, "" $\cos^{-1}$, ", and " $\tan^{-1}$ " functions instead.

Review Questions

3-4) Students are likely to stop when they have found the one solution within the restricted range, and forget that they are looking for solutions between 0 and 2π rather than just between 0 and π.

Solving Trigonometric Equations Using Inverse Functions

In-Text Examples

3) The most likely error here is for students to try "unwrapping" the right-hand side of the equation in the wrong order, for example by taking the inverse cosine before dividing by A, or by subtracting ϕ before it is properly isolated. This sort of error is often more likely when working with expressions that have a large number of variables, as students can get a little bit lost when they are not working with concrete numbers.

Review Questions

1) This problem is subject to the same error as example 3 above; also, students who do not recognize that the problem relies on an angle sum identity will approach it wrong from the beginning and get an answer that makes no sense, if they get any answer at all.

Solving Inverse Equations Using Trigonometric Identities

In-Text Examples

1) This example is much trickier than it seems at first, because of the double angle. Watch out for students forgetting to account for the 2 and just solving for $\cos(\sin^{-1} x)$, which is $\sqrt{1-x^2}$.

Review Questions

1) Students may get their double- and half-angle identities mixed up here, or may forget that if they are applying a double-angle identity to $\sin 2\theta \cos 2\theta$, they will end up with $\sin 4\theta$ rather than $\sin 2\theta$.

2) (Note: 10 feet is actually the distance from one end to the other, rather than the width of the one end.)

Students may think they don't have enough information to solve this problem, but they do if they express the legs of the triangles as $\sin\theta$ and $\cos\theta$; then they can set up an equation for the area of one end (and thus the volume) in terms of θ. (They may need calculators, though, to find the equation's maximum value.)

1.5 Triangles and Vectors

The Law of Cosines

In-Text Examples

1) A common error when working with the Law of Cosines is to forget to subtract, rather than add, the last term ($2ab\cos C$). Also, since students will be using their calculators for many of these problems, they may need to be reminded to check that they are in the right mode. (Degrees, rather than radians, will almost always be the mode used for problems dealing with angles in triangles.)

2) In cases where students know all three side lengths of a triangle and are trying to find one of the angles, they may fail to plug the side lengths into the formula in the correct places. A mnemonic that may help is "the length of the side *opposite* the angle we are trying to find goes on the *opposite* side of the equation from the other two side lengths."

Review Questions

4) Students may forget here that 23.3° is the measure of $\angle ABD$ rather than $\angle ABC$.

6) Once students have found the correct measure of the angle or side, they may well forget to subtract it from the given measure to tell how far off the given measure is.

10) Part a should read "If the tee is 329 yards . . .." as in the diagram.

On part b, students may be a little confused about how to set up the new diagram. Basically, it should look like the previous diagram, except with "98 yds" in place of "235 yds" and "3°" in place of "9°."

12) The instruction "use some right triangle trig" may give some students the idea that the triangle described in the problem is a right triangle. Other students may need to be reminded of the formula for the area of a triangle. Others may overcomplicate the problem by trying to find the measures of more than one angle of the triangle when just one will do.

14) Some students may try to treat this quadrilateral as a parallelogram and find the area based on that formula, instead of finding the areas of the two triangular subsections separately.

15) Students are liable to think that 17° or 82° is the interior angle of the triangle that lies between the two sides of 20m and 4m length, when it is really the exterior angle (and thus the supplement of the interior angle).

More rarely, students may think that 17° is the measure of one angle of the triangle and 82° is the measure of another of the angles, instead of using 17° as an angle measure in part a and 82° as the measure of the same angle in part b.

16) Since the length of AB is not given, students may try to subtract 5 cm from one of the three other lengths that are given in the diagram. Impress upon them that this is a two-part problem: first they need to find the length of AB when $\angle AEH$ is 120°, and then they must subtract 5 cm from that length and apply the Law of Cosines in reverse to find the new measure of $\angle AEH$.

Area of a Triangle

In-Text Examples

2) It's very easy, when using Heron's Formula, to slip up and forget to divide by 2 when calculating s. It's also easy to overlook the factor s under the square root sign, as it is somewhat dwarfed visually by the other three factors.

Review Questions

1) In parts b through d, students may think they are supposed to find the areas of the right triangles drawn in with dotted lines, instead of the oblique triangles drawn with solid lines—or they may think, because of the right triangles, that they should use the "$\frac{1}{2}bh$" formula every time.

Also, they may try to actually find the areas instead of merely stating which formula to use to find them, but that isn't so bad in this case since it's what the next question asks them to do anyway. This also applies to problems 3 and 4.

2) Rounding off too early may get some students in trouble here; remind them if necessary not to round off until the very last step so that roundoff errors don't accumulate.

5) On this problem, students may be tempted to plug in 375 feet as the height of each triangle in the formula $\frac{1}{2}bh$, instead of realizing that 375 feet is the length of the diagonal legs of each triangle and that they must use Heron's formula to calculate the areas. Also, they may forget to quadruple the area they find for one triangle to get the total area for all four.

6) Students may need to be reminded here that the contractor will have to buy a whole number of bundles.

7) 14955.6 square yards is the new area, but the question asks for the difference between the new area and the old.

9) After plugging the area of the triangle into the $\frac{1}{2}bh$ formula, students will find that the length of the triangle's base is approximately 39.34 units; they may think this is their final answer and fail to subtract 14.4 from it to get 24.94.

Additional Problems

1) Use the identity from problem 10 to prove the Pythagorean Theorem; that is, show that $f^2 = d^2 + e^2$ for any right triangle DEF where f is the hypotenuse.

Answers to Additional Problems

1) Start with the identity from problem 10:

$$d^2 + e^2 + f^2 = 2(ef \cos D + df \cos E + de \cos F)$$

Now if F is a right angle, $\cos F = 0$, so that term drops out and we have:

$$d^2 + e^2 + f^2 = 2(ef \cos D + df \cos E)$$

Drawing the triangle will show us that $\cos D = \frac{e}{f}$ and $\cos E = \frac{d}{f}$, so we can substitute:

$$d^2 + e^2 + f^2 = 2\left(ef \cdot \frac{e}{f} + df \cdot \frac{d}{f}\right)$$

Canceling out the fs:

$$d^2 + e^2 + f^2 = 2(e^2 + d^2) = 2(d^2 + e^2)$$

And subtracting $(d^2 + e^2)$ from both sides:

$$f^2 = d^2 + e^2$$

The Law of Sines

Review Questions

1) Part c is a trick question; it is neither AAS nor ASA but AAA, which we haven't covered yet.

2) Because of problem 1c, students may assert that the triangles in the chart don't all have anything in common, or don't have anything in common except that we know at least two angles. This is technically correct, but the problem is really trying to get at what the ASA and AAS cases have in common, ignoring the AAA case.

Students also shouldn't be penalized for answering that in both cases we know two angles (or can find three angles) and one side.

3) Again, part c can't be solved, so don't let students get stuck on it. (You may want to stress the "if possible" part of the instructions before they get started.)

6) Students may mix up the ASA and SAS cases here, leading them to mix up the Law of Sines and Law of Cosines cases in turn. Also, they may try to actually solve the triangles instead of just stating how they would solve them.

7) The x we are solving for here is the "other half" of the base of the triangle.

8) After doing all the steps needed to find the various distances, students may forget to subtract the old distance from the new distance, or may forget to calculate the time the extra distance took as their final step.

9) Students may forget to have the driver start at the warehouse rather than at stop A, or may forget that she needs to get back to the warehouse after stop C; in other words, they may forget to add in that extra 1.1 miles of distance either or both times.

Also, they may label the angles wrong, for example labeling the 103° angle at stop B as the exterior angle rather than the interior angle of the triangle; this is an easy mistake to make here, since they may unconsciously be thinking of the angles the two other streets make with First Street as both being in "standard position" because First Street is horizontal.

Finally, they may forget to add in the 2 minutes for each package when calculating the time, or may add 2 minutes just once instead of three times. Or, they may get the time constraints backwards, thinking of the driver as leaving at 10:00 and trying to calculate when she will get back, instead of calculating when she must leave to get back by 10:00.

10) There is more than one way to set up this problem, depending on how one interprets the descriptions given. Students therefore may not get the answer given, but should not be penalized if they can show that they have set up the problem in a way that seems reasonable.

The Ambiguous Case

Review Questions

3) Students may approach this problem in reverse, by rewriting the given equation as $\frac{a}{c} - \frac{c}{c} = \frac{\sin A}{\sin C} - \frac{\sin C}{\sin C}$ and going from there, ending up with the Law of Sines as their final step. This is an equally valid solution.

4) Students may have a hard time coming up with an appropriate set of sides and angle if they try picking the sides first; they will have much better luck if they think of starting with an angle A and a side b and then finding a side a that is less than b and greater than $b \sin A$. Even then, though, this will only work if they pick an acute A, although it may not be immediately obvious why. (Drawing a picture may make it clearer: if A is the biggest angle, then a must be the biggest side and certainly can't be smaller than b.

5) Students may think there is only one value of A in each case, rather than a whole range of values.

6) Solving this problem without using the Law of Sines at all is actually possible, and students should probably not be penalized for doing so.

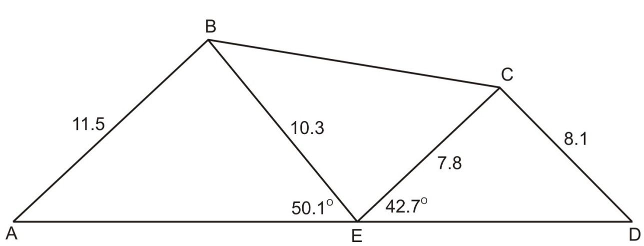
7) Students may get stuck if they try to find all the sides and angles in the order specified; they should be encouraged to find them in whatever order they can. Also, make sure they realize that it's $\angle ABC$ that measures 109.6°, rather than any of the other angles whose vertices are at B.

8) Some students may jump to the conclusion that the triangle shown is a right triangle, simply because it looks like one.

10) The last question can't actually be answered using the information given.

Additional Problems

1) In the figure below, $AB = 11.5, BE = 10.3, EC = 7.8, CD = 8.1, \angle AEB = 50.1^\circ$, and $\angle CED = 42.7^\circ$.



Find the following, to the nearest tenth of a unit:

a) $\angle BEC$

b) BC

c) $\angle EBC$

d) $\angle ECB$

e) $\angle BAE$

f) $\angle ABE$

g) AE

h) $\angle EDC$

i) $\angle ECD$

j) ED

Answers to Additional Problems

1) a) 87.2° (because $\angle AED$ is a straight angle)

b) 12.6 (by Law of Cosines)

c) 38.2° (by Law of Sines)

d) 54.7° (by Law of Sines)

e) 43.4° (by Law of Sines)

f) 86.5° (by Triangle Sum Theorem)

g) 15.0 (by Law of Sines)

h) 40.8° (by Law of Sines)

i) 96.5° (by Triangle Sum Theorem)

j) 11.9 (by Law of Sines)

General Solutions of Triangles

Review Questions

3) On part d, students may think that, as on part e, we are still missing side c and angle C. However, the fact that there is no solution possible to this triangle actually means that there is no such triangle, so technically we are not "missing" that last side and angle measure because the side and angle themselves do not exist.

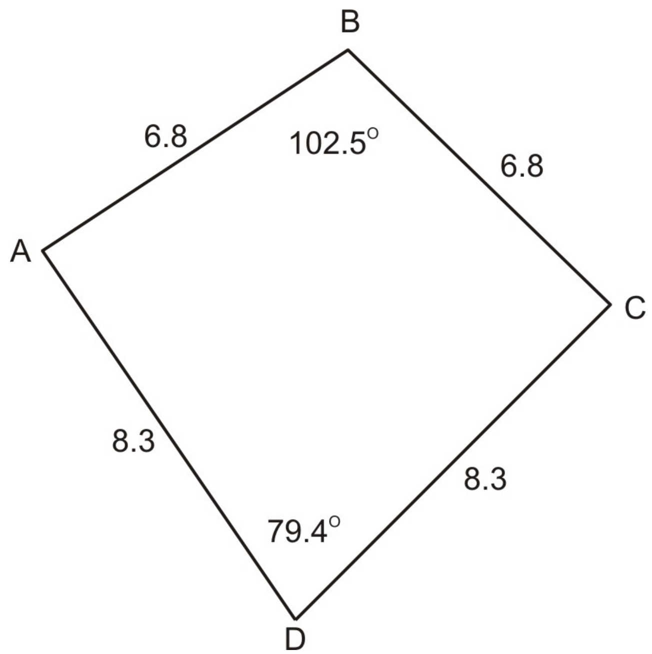
5) Students may need to be reminded that a rhombus has all four sides the same and that both pairs of opposite angles are congruent. They may also need to be reminded that they can use Heron's formula to find the area of half of the rhombus, and they may forget to double that area to find that of the whole rhombus.

6) The likeliest error here, once students realize they need to divide the pentagon into triangles to get anywhere with it at all, is for them to divide it into triangles they can't solve. This will happen if they connect the vertices whose angles are already known, dividing them up into unknown angles. Instead, they need to connect the three vertices whose angles aren't given; then they will have triangles they can solve.

7) There isn't enough information given to solve the triangle; you may need to supply a couple of angle or side measures from the answer key so that students can find the rest.

Additional Problems

1) Find the area of the quadrilateral below, to the nearest hundredth.



2) Find the missing angle measures in the above quadrilateral, to the nearest tenth.

Answers to Additional Problems

1) First, divide the quadrilateral into two triangles by drawing diagonal AC. Then the area of each triangle can be found with $K = \frac{1}{2}bc \sin A$.

The area of $\triangle ABC$ is 22.57, and the area of $\triangle ADC$ is 33.85. The total area of the quadrilateral is therefore 56.42.

2) The fact that $ABCD$ is symmetrical is what makes this problem possible to solve. Divide $ABCD$ into two triangles by drawing diagonal BD; these two triangles are congruent by SSS, and so each of them has area 28.21 (half of the area of $ABCD$). The area formula $K = \frac{1}{2}bc \sin A$ can now be used to find the missing angle in each triangle.

Both missing angles, $\angle A$ and $\angle C$, measure 88.5°.

Vectors

In-Text Examples

1) Stress that the distance formula is just a version of the Pythagorean Theorem: the magnitude of the vector can be thought of as $\sqrt{(\text{difference between x} - \text{coordinates})^2 + (\text{difference between y} - \text{coordinates})^2}$.

Review Questions

1) The answers given in the text include the vector directions with respect to $\vec{m}$—that is, the direction of each resultant vector is expressed as the angle the vector makes with $\vec{m}$—so students will get the same answers as those in the text if they too express their resultant vectors in terms of $\vec{m}$. If you don't specifically tell them to do this, though, they may express them in terms of $\vec{n}$ instead, in which case the angles they give as their answers should be the complements of the angles given in the book. (Either way is correct in the absence of more specific instructions.)

2) Students can use either the triangle or parallelogram method here. Since they will need to use a ruler and protractor, make sure their copy of the text is the right size or they will get the wrong answers. (It may be worth double-checking by measuring the lengths of the vectors $\vec{a}$ through $\vec{d}$ as they appear on the page.)

Also, note that the diagrams on the first two lines of the chart are not to scale; students will have to re-draw the vectors to get the correct magnitude and direction.

And finally, they will be measuring angles from the horizontal when there isn't a horizontal line drawn in the text, so they will have to estimate them as best they can; don't be too strict about their getting the angles exactly right.

3) This is actually only true if $\vec{a}$ and $\vec{b}$ have the same or opposite direction.

4) The direction angle, 4.6° NW, can also be expressed as an angle of 94.6° in standard position.

8-10) Students will have to draw some fairly large vectors to solve these problems, and they will still need to scale them down in order for them to fit on one page. As a result, they may not be able to get their answers accurate to the nearest tenth or even the nearest whole unit. (They won't have this problem if they use the Law of Cosines and the Law of Sines instead of drawing the vectors, but since that technique hasn't been covered yet, it probably won't occur to them to use it and they certainly shouldn't be expected to.)

Also, on problems 8 and 10, note that the angle between the two vectors in each case is the angle between them when they are placed tail-to-tail. This makes the parallelogram method fairly easy to apply, but students who use the triangle method instead may think that the angle given is the angle between the vectors when they are placed tip-to-tail, and will get the wrong answer as a result.

Component Vectors

In-Text Examples

5) Simply multiplying the coordinates by 2.5 without translating first yields a directed segment from $(10, 17.5)$ to $(30, 27.5)$. This answer isn't wrong, as it has the same magnitude and direction as the one from $(4, 7)$ to $(24, 17)$, but you may want to make sure students understand that the answers are equivalent.

Review Questions

2) Students may try to solve these problems by translating the vectors to the origin and reading off the coordinates of the new terminal point, or they may try to solve them by finding the difference between the $x-$coordinates of the two points and the difference between the $y-$coordinates of the two points. It may be worth pointing out that these two methods are in fact mathematically equivalent; both involve performing essentially the same operations and both yield the same answer.

8) This question may confuse students a bit; some of them may try to solve it as if they were given two components and asked for the resultant vector, while others may try to solve it as if they were given the resultant and one component and asked for the other component. Since the text isn't entirely clear on which one is actually being asked for, it's best to accept any answer that a student can justify.

9) This question is similarly ambiguous; as a result, students may get 11.5° rather than 11.3° as their answer, and this should be accepted.

10) You may need to clarify the notation here: AB means simply the directed line segment from point A to point B.

Additional Problems

1) Find the single ordered pair that represents $\vec{a}$ in each equation if you are given $\vec{b} = (1,2)$ to $(3,8)$ and $\vec{c} = (3,3)$ to $(5,2)$.

a) $\vec{a} = \frac{1}{2}\vec{b}$

b) $\vec{a} = -5\vec{c}$

c) $\vec{a} = \vec{b} - \vec{c}$

d) $\vec{a} = 3\vec{c} + 2\vec{b}$

Answers to Additional Problems

1) (By translating $\vec{b}$ and $\vec{c}$ to the origin, you can represent each of them as a single ordered pair, which makes it easier to represent $\vec{a}$ as a single ordered pair.)

a) $(1,3)$

b) $(-10,5)$

c) $(0,7)$

d) $(10,9)$

Real-World Triangle Problem Solving

Review Questions

2) To solve this problem, we need to assume that the height of the canyon wall on the hiker's side is the same height as on the opposite side. (Most students will assume this anyway, but those who don't should be told to.)

5) Some few non-pool-playing students may need to be told which ball is which in the diagram, and may need to be told that the pockets are at the corners of the table.

7) Some students may not immediately see that they need to find the area of the triangle between the three docks. Others may find the area and forget to multiply by 5.2×10^{13} to find the number of bacteria.

8) Students may jump to the conclusion that $37°$ is the measure of the angle from tower B to tower A to the fire (it's actually the angle's complement.) They may also jump to the conclusion that the angle from tower A to tower B to the fire is a right angle.

10) Students may think the angle between the first part of the trip and the second part of the trip is $34°$. Instead, they need to think in terms of vectors; they should draw the two halves of the trip as individual vectors with their separate headings, and then find their resultant.

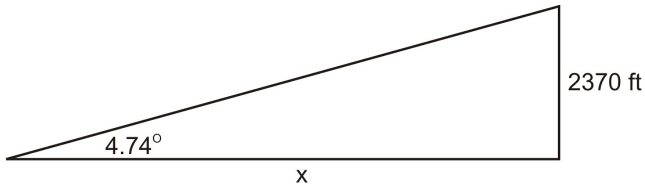
Additional Problems

1) a) A ship approaches close enough to shore to spot a famous lighthouse which is known to be 180 feet tall, and which stands at the top of a $2200-$foot cliff. The captain, looking from a point 15 feet above the water, observes that the angle of elevation to the top of the lighthouse is $4.74°$. How far is the ship from the lighthouse, to the nearest foot? To the nearest tenth of a mile? (5280 feet = 1 mile.)

b) The lighthouse keeper notices that the ship appears to be headed toward a dangerous reef. The reef is known to be 2.8 miles from the lighthouse, and the lighthouse at this moment makes an angle of 30.8° with the ship and the reef. How close is the ship to the reef?

Answers to Additional Problems

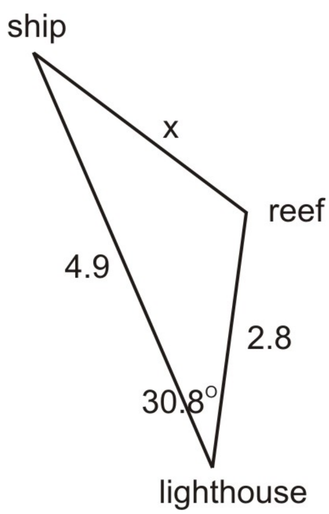
1) a)



$$\tan 4.74^\circ = \frac{2370}{x} \quad x = \frac{2370}{\tan 4.74^\circ} \quad x = 28,522$$

The ship is 28,522 feet, or 4.9 miles, from the lighthouse.

b)



Using the Law of Cosines:

$$x^2 = (4.9)^2 + (2.8)^2 - (2 \times 4.9 \times 2.8 \cos(30.8^\circ))$$
$$\therefore x \approx 2.9$$

The ship is 2.9 miles from the reef.

1.6 Polar Equations and Complex Numbers

Polar Coordinates

Polar Coordinates

In-Text Examples

2) You may want to stress that if $(4, 120°)$ is one way of representing the given point, $(-4, -120°)$ is *not* another way. In general, if we change the sign of the $r-$coordinate, we cannot simply change the sign of the $\theta-$coordinate to keep the point the same.

(Here's part of the reason why: When we change the sign of the $r-$coordinate, we're actually rotating the point 180° about the origin—or reflecting it across both the $x-$ and $y-$axes, which is the same thing. When we change the sign of the $\theta-$coordinate, we're reflecting the point across the $x-$axis. So if we change the signs of both coordinates, those two transformations combined add up to one reflection across the $y-$axis. Explaining this is optional, though; the important part is that changing the signs of both coordinates doesn't get us back to the same point.)

Review Questions

2) Students' answers should include the original coordinates $\left(-4, \frac{\pi}{4}\right)$ rather than the third pair of coordinates given in the answer key.

Sinusoids of One Revolution

In-Text Examples

5) Students may have trouble graphing looped limaçons like this if they don't include enough values in their table. Also, it may be hard for them to keep track of which order the points should be connected in, especially when the $r-$values become negative.

Review Questions

1) All the curves here are limaçons, but that answer is not sufficient.

2) The rose will have n petals if n is odd and $2n$ petals if n is even, but that may not be apparent just from the two examples given. Students shouldn't get hung up on trying to find a more specific relationship than the one given in the answer key.

Applications, Trigonometric Tools

In-Text Examples

2) Students may think that the total distance θ they must plug into the area equation is $\frac{\pi}{3}$ rather than $\frac{2\pi}{3}$, because of the $\frac{\pi}{3}$ coordinate used earlier. The graph should help them avoid this mistake, though.

3) Any three roses will do, not just the three shown. Creating a quilt can't easily be done on just one set of axes, though; it will need to be done by copying one graph several times onto a sheet of paper.

Polar-Cartesian Transformations

Polar to Rectangular

Review Questions

1) It's common to get the equations $x = r\cos\theta$ and $y = r\sin\theta$ backwards when solving a problem, and think that $y = r\cos\theta$ and $x = r\sin\theta$; also, it's common to think of $\tan\theta$ as being equal to $\frac{x}{y}$ instead of $\frac{y}{x}$. Watch for these errors on all problems of this type.

Also, if students use calculators on this problem they may forget to switch modes on part B.

Rectangular to Polar

In-Text Examples

1) You may need to stress the part about adding π to the arctangent when x is less than 0, so that students

remember it when solving future problems.

Review Questions

1) Students working in degrees instead of radians will get $A(5.39, 111.73°)$ and $B(6.40, -38.39°)$ as their answers.

Conic Section Transformations

In-Text Examples

1) Mixing up the x's and y's is a common error when dealing with conics. When working with parabolas in particular, students may default to treating every one (if they don't sketch it out carefully first) as if it opened up or down instead of left or right.

5) You may want to point out that plugging in a negative value for a doesn't yield a circle with a negative radius (what would that even look like?), but rather yields a circle of radius a that's on the "other" side of the origin—left instead of right for a cosine equation, and down instead of up for a sine equation. (This makes sense if you picture what happens to a graph like this as a decreases toward zero and finally passes zero—the circle shrinks closer and closer to the origin until it passes through the origin and comes out the other side. It also makes sense if you remember that multiplying $r-$values by -1 is the same as rotating the graph 180°.)

Review Questions

1) Students may try graphing the equation to prove it is a parabola, but this is more time-consuming and prone to error than the method described in the answer key.

2) Again, watch for students mixing up x and y, or a and b, or h and k.

Applications, Technological Tools

Rectangular Form or Polar Form

3) You may need to explain that the sun is at one focus of the ellipse, so the perihelion is the distance from F to P and the aphelion is the length of the major axis minus the perihelion.

Systems of Polar Equations

Graph and Calculate Intersections of Polar Curves

In-Text Examples

3) Students may be very confused by the idea that $(0, 0)$ and $\left(0, \frac{\pi}{2}\right)$ represent the same point—it makes sense that adding 2π to the $\theta-$coordinate of a point yields another name for the same point, but it doesn't make sense that adding $\frac{\pi}{2}$ would do so. In fact, the only reason it works in this case is that the $r-$coordinate of the point is 0. If we rotate through any angle whatsoever, but then go 0 units from the origin, we end up at the origin—so any pair of coordinates where the $r-$coordinate is 0 is just one more way of naming the point $(0, 0)$, no matter what the $\theta-$coordinate is.

4) It may seem a little strange that the polar and rectangular coordinates for the two points of intersection are exactly the same. However, this is generally true for any point whose $\theta-$coordinate is 0: the $y-$coordinate in rectangular form will also be 0, and the $x-$coordinate will be the same as the $r-$coordinate. Doing the conversion algebraically will confirm this, and it can be handy to know.

Review Questions

1) The graph of these equations is deceptive; it certainly looks as though they intersect three times rather

than just one. But tracing the graph of $r = 3\ \sin\theta$ will show that at two of the apparent intersection points, the $r-$value is actually negative and so is not the same as the $r-$value of $\sin 3\theta$ at the same point. Solving the system algebraically will also show that there is really only one solution.

Equivalent Polar Curves

In-Text Examples

1) Part b should read $5\cos(-90)$ rather than 2; also, the number 90 may make it seem like we are working in degrees here, but we're actually still working in radians. (The equations are basically equivalent to $r = 2.24$ and $r = -2.24$ respectively. If we were working in degrees, those same equations would be equivalent to $r = 5$ and $r = -5$.)

In general, the graph of $r = a$ is equivalent to the graph of $r = -a$.

Review Questions

2) There's a small chance students will get mixed up here and substitute θ for π in their calculations, because they are used to r being expressed in terms of θ. In fact, the equations are simpler than they look; since $\frac{\pi}{3}$ is a constant, the whole right-hand side of each equation is a constant (it works out to simply 5.5), so the graph is just a circle centered at the origin.

Applications, Technological Tools

In-Text Examples

You might want to point out in the example here that we are concerned with both the "real" points of intersection and the "apparent" ones, since this is a real-world problem where we are concerned with the shape of the graph and not just the actual number values of the points.

Imaginary and Complex Numbers

Recognize

In-Text Examples

A very common error when simplifying square roots is to pull numbers or expressions out from under the radical sign without actually taking their square roots; this tends to happen when there is more than one expression under the radical sign to deal with. For instance, in example 1a, students might express the answer as $16i$ rather than $4i$; or in example 2c, they might express the answer as ix rather than $i\sqrt{x}$.

Review Questions

1) On parts c and d, students might make the error they were warned against in the text and come up with positive 15 and 35 as answers.

Standard Form of Complex Numbers $(a + bi)$

In-Text Examples

2) Students are likely not to fully grasp that they can treat the real terms and the imaginary terms completely separately, and that this equation therefore can really be thought of as two separate equations, one which can be solved for x and the other for y. They may also forget momentarily that i is not a variable, and start vaguely trying to solve for it as well as for x and y, or may just get the idea that there is one equation here with three variables and hence not enough information to find a solution.

Additional Problems

1) Find the conjugate of each complex number:

a) $3 + 0i$

b) 10

Answers to Additional Problems

1) a) 3–0i or simply 3

b) 10–0i or simply 10

The Set of Complex Numbers

Additional Problems

1) What sets does each number belong to?

a) $\sqrt{25}$

b) $\frac{\pi}{3}$

c) $\sin \frac{\pi}{3}$

d) $\cos \frac{\pi}{4}$

Answers to Additional Problems

1) a) complex, real, rational, integer, whole number, natural number

b) complex, real

c) complex, real, rational, fraction

d) complex, real

Complex Number Plane

Review Questions

1) The absolute values given are for points A and E; students may have chosen other points instead, so those answers are also acceptable.

Operations on Complex Numbers

Quadratic Formula

In-Text Examples

Sign errors are very likely to occur when working with the quadratic formula.

Review Questions

1) Students may forget that one side of the equation must equal zero before they can apply the quadratic formula, so they may try to simply read off the coefficients from the left-hand side of the equation.

Additional Problems

1) a) What does the discriminant of $-x^2 + 6x - 9$ tell you about its root(s)?

b) Calculate the root(s).

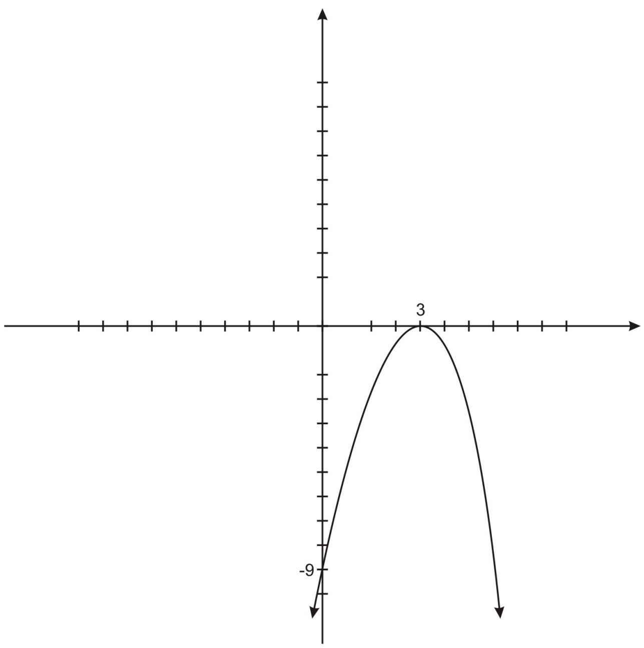
c) Graph the equation.

Answers to Additional Problems

1) a) The discriminant is zero, so the function has one repeated real root.

b) The one root is 3.

c)



Sums and Differences of Complex Numbers

Review Questions

1) The answers students get from adding the vectors graphically may not be as precise as the answers they get from adding the numbers algebraically, so they may not quite match each other. This is fine as long as they are reasonably close.

Products and Quotients of Complex Numbers (conjugates)

In-Text Examples

1) This is another likely place for sign errors: when multiplying complex numbers and simplifying the answer, students are quite likely to convert i^2 to 1 instead of 1, and so simply drop the i^2 altogether without changing the sign of its coefficient.

Trigonometric Form of Complex Numbers

Trigonometric Form of Complex Numbers: Steps for Conversion

In-Text Examples

3) After learning to work with square roots of negative numbers, students may get a bit confused when finding r based on negative values of x and y; they might forget that squaring those negative values should yield positive numbers, and that they should not be trying to take the square roots of any negative numbers to find r.

Review Questions

2) Graphing the number on a rectangular coordinate graph is one option students may take, although that requires expressing it in standard form first and is slightly harder than graphing it in polar form.

Additional Problems

1) Express the sum of $3+6i$ and $9-2i$ in polar form.

Answers to Additional Problems

1) The sum of the two numbers is $12+4i$, so in rectangular form $x=12$ and $y=4$.

$$\begin{aligned} r &= \sqrt{x^2+y^2} \\ &= \sqrt{12^2+4^2} \\ &= \sqrt{144+16} \\ &= \sqrt{160} \\ &= 4\sqrt{10} \end{aligned}$$

$$\begin{aligned} \tan\theta &= \frac{y}{x} \\ &= \frac{4}{12} \\ &= \frac{1}{3} \\ \theta &= \tan^{-1}\frac{1}{3} \\ &\approx 18.43^\circ \end{aligned}$$

So in polar form the number is $4\sqrt{10}$ cis 18.43°.

Product and Quotient Theorems

Using the Quotient and Product Theorem

In-Text Examples

1) When applying the product rule, probably the easiest error for students to make is to get confused about whether to add the $r-$values and multiply the $\theta-$values or vice versa. Students may also try to add them both or multiply them both.

(Similar errors will occur when applying the quotient rule.)

3) Students may get confused when the numbers to be multiplied or divided are presented in rectangular rather than polar form; they may try to treat the real parts like r-coordinates and the imaginary parts like $\theta-$coordinates, rather than converting to polar form first so they can do the calculations properly.

Additional Problems

1) Find the quotient: $(7+2i\sqrt{2}) \div (\sqrt{3}-4i)$. Express your answer in rectangular form rounded to two decimal places.

Answers to Additional Problems

1) First, convert to polar form:

$$r_1 = \sqrt{x^2+y^2} = \sqrt{7^2+(2\sqrt{2})^2} = \sqrt{4+8} = \sqrt{57}$$

$$r_2 = \sqrt{x^2+y^2} = \sqrt{\sqrt{3}^2+(-4)^2} = \sqrt{3+16} = \sqrt{19}$$

$$\frac{r_1}{r_2} = \frac{\sqrt{57}}{\sqrt{19}} = \frac{\sqrt{19.3}}{\sqrt{19}} = \frac{\sqrt{19}\sqrt{3}}{\sqrt{19}} = \sqrt{3}$$

$$\theta_1 = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{2\sqrt{2}}{7} \approx 22.002^\circ$$

$$\theta_0 = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{4}{\sqrt{3}} \approx 66.587^\circ$$

$$\theta_1 - \theta_2 \approx -44.585^\circ$$

So the quotient in polar form is $\sqrt{3}$ cis $-$ 44.585°.

Converting back to rectangular form:

$$x = \sqrt{3}\cos(-44.585^\circ) \approx 1.23$$

$$y = \sqrt{3}\sin(-44.585^\circ) \approx -1.22$$

So the final answer is $1.23 - 1.22i$.

Powers and Roots of Complex Numbers

De Moivre's Theorem

In-Text Examples

1) As with the product theorem, students may get confused about which coordinate to multiply by n and which one to raise to the n^{th} power when applying De Moivre's Theorem. Also, even after writing down "$\cos n\theta$" and "$\sin n\theta$," they may still try to find $n \ \cos\theta$ and $n \ \sin\theta$ instead.

Review Questions

1) As with the product theorem once more, students may forget to convert to polar form before applying De Moivre's Theorem...

2) ...or may forget to apply De Moivre's Theorem before converting to rectangular form.

nth Root Theorem

Additional Problems

1) Find all the fourth roots of 81.

2) Find all the sixth roots of $64i$.

Answers to Additional Problems

1) $3, 3i, -3, -3i$

2) The principal root is $2 \text{ cis} \frac{\pi}{6}$, or $2\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$; the five other roots are $2 \text{ cis} \frac{\pi}{2}, 2 \text{ cis} \frac{5\pi}{6}, 2 \text{ cis} \frac{7\pi}{6}, 2 \text{ cis} \frac{3\pi}{2}$, and $2 \text{ cis} \frac{11\pi}{6}$.

Solve Equations

Review Questions

1) This problem is an easy place to make sign errors.

Applications, Trigonometric Tools: Powers and Roots of Complex Numbers

In-Text Examples

1) Students may actually be able to skip the step of calculating the three roots if they realize that the roots are evenly spaced about a circle and are able to figure out how to graph them based on that knowledge. This probably shouldn't be penalized, as it demonstrates understanding of the principles behind n^{th} roots.

2) The bit about using the Pythagorean Theorem and polar coordinates is somewhat of a red herring; those can be used to find other values in the given diagram, but to find L, students need only use the Law of Cosines.